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COMPARING THE ALGEBRA PROBLEM SOLVING PROCESSES OF STUDENTS AND MATH PRACTITIONERS

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ABSTRACT

The United States Department of Education's National Assessment of Educational Progress reports that most US students perform below grade level in math. Much research has been conducted on how to improve math performance in students. The present study compares the algebra 1 problem solving processes used by students to those used by math practitioners, professionals who use math as part of their jobs. Eleven algebra 1 students and eight math practitioners were given a set of 20 algebra 1 word problems. Each was asked to solve the problems while a protocol of his or her problem solving processes was recorded. The protocols were analyzed to determine commonalities and differences in problem solving processes both within the practitioner and student groups and between them. Results suggested that students were fairly homogeneous and solved problems algebraically by creating and solving formulas, employing procedural processes as exemplified by John Anderson's ACT-R framework (1982). Practitioners were more diverse in their approaches and spent more time analyzing the problems up front and used their conceptual analysis to generate problem solving processes that often were simpler than those employed by students and relied heavily on heuristics and pattern recognition, akin to subject matter experts reported in Leddo et al.'s research (1990). Implications for mathematics teaching approaches to build these types of practitioner problem solving skills are discussed.

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Introduction

The United States Department of Education reports in its National Assessment of Educational Progress (2020) that a majority of US students perform below grade level in math. This presents a national challenge, given the centrality of math in STEM (science, technology, engineering and math) fields and the necessity for numeracy skills in daily life. Part of the reason for low performance in math may be that math skills degrade after they have been learned, particularly as the difficulty of the mathematics content increases. For example, Leddo at al. (2021) found that within a few months of learning the subject matter, average student performance is about 85% for fifth grade math concepts, 65% for algebra 1 concepts, and 46% for algebra 2 concepts.

In addition to the quantitative factors (degree of forgetting) that may influence how well students perform in math, there may be qualitative factors as well, namely how students conceptualize mathematical concepts and problem solving processes. How students think about math and approach solving math problems may offer insight into the problems they have and potential remedies for improving performance in math.

One area of study that may offer guidance on how to boost student achievement is expert problem solving. Expert problem solving research has enjoyed a long history, fueled in large part by early attempts to create expert systems. A typical paradigm was to present experts with representative problems to solve, ask them to solve the problems while thinking out loud about the processes they were using, recording the problem solving protocol and then analyzing the protocols to extract and generalize the process (cf., Ericsson and Simon, 1984).

This type of research led to a rich body of data about what knowledge experts have and how experts solve problems in a variety of subject areas, including academic areas such as mathematics. For example, Newell and Rosenbloom (1981) found that experts tend to chunk individual pieces of knowledge together to form larger units, thus making them easier to remember and use. A novice (someone less experienced in a subject) might view solving an equation as a sequence of individual steps. An expert may view the same process as "reverse order of operations" or "SADMEP" (subtraction, addition, division, multiplication, exponents, parentheses), whereby a single concept represents the entire problem solving process.

Another factor differentiating experts and novices in algebra problem solving is in how problems are categorized. Experts tend to classify problems based on solution strategies, whereas novices tend to categorize problems based on the words or objects mentioned in the problems (Shoenfeld and Herrman, 1982).

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Perhaps most relevant to the present study is the work done to investigate how people develop expertise. One of the most influential frameworks in this area is the ACT-R framework developed by John Anderson (1982). ACT-R posits that people start by learning facts and then proceduralize those facts into actual problem solving processes. An example of this in algebra might be solving a two-step equation. A student may learn that s/he needs to isolate the variable in order to find its value. This is a fact, but one that does not tell the student what s/he actually needs to do. The student may then learn topics like additive inverse and multiplicative inverse that enable such isolation of the variable to occur. The student learns a procedure for isolating the variable by first applying the additive inverse to both sides of the equation and then multiplying the multiplicative inverse of the variable's coefficient to both sides of the equation. Once a student can perform these operations, s/he is said to have proceduralized the factual knowledge and can now apply that knowledge procedurally to solve problems. According to the ACT-R framework, a person can be considered an expert when s/he has attained this level of procedural knowledge.

Leddo et al. (1990) took the ACT-R theory one step further. Leddo et al. found that experts did have a great deal of procedural knowledge, but that procedural knowledge alone was not sufficient to make someone an expert. Instead, experts had abstract causal/conceptual knowledge that captured the underlying rationale for why problem solving strategies worked and this knowledge allowed experts to apply different variations of a problem solving procedure to a nuanced situation or innovate in a completely novel situation. Leddo et al. noted that the ACT-R framework was necessarily incomplete as it would require experts to have explicit procedures for each problem type the expert encountered and would not explain how experts could find solutions to novel situations they never previously encountered. Research shows that experts contain knowledge over and above procedural knowledge that allows them to perform in novel situations (Schwartz, Bransford and Sears, 2005).

Other differences emerged from the Leddo et al. study. Experts spent considerable amount of time conducting upfront analyses of problems they are solving. This time involved goal setting, information gathering, defining variables and unknowns and determining problem solving strategies. This finding was also echoed in the work by Atman et al. (2007) and Hurwitz et al. (2014).

The extensive research literature on expert problem solving indicates that understanding how experts solve problems can be very useful to educators concerned with improving mathematics performance in students. For the purposes of the present study, we believe that it is not necessary to get students to emulate the problem solving of experts. As researchers in the field of expert

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knowledge widely acknowledge, it can take 10 - 20 years for a person to become an expert. Therefore, trying to get students to emulate experts may be too ambitious for most elementary and secondary school students.

We chose instead to look at math practitioners, those who do math as part of their everyday jobs. Such people could be engineers, accountants, statisticians, etc. It may not be necessary that these people be actual experts. However, for them to be successful at their jobs, they need to be proficient problem solvers. Becoming proficient problem solvers seems like an achievable and appropriate goal to have for students regarding their math problem solving. Accordingly, the goal of the present study is to understand and compare the problem solving processes used by students and math practitioners to see what skills the practitioners have that students do not and that might be beneficial for the students to have. This could lead to the development of teaching methods that would enable students to improve their problem solving performance.

In the present study, algebra 1 was chosen as a testbed for mathematics problem solving. Both algebra 1 students and math practitioners were given word problems to solve and their problem solving processes were recorded and compared.

Method

Participants

Eleven Algebra 1 students and eight math practitioners participated in the present study. All were recruited from the Northern Virginia area in the United States. Math practitioners were defined as people who used math as part of their jobs. Of the eight math practitioners, one is an actuary, one is an accountant, one is a quantitative analyst, one is a civil engineer, two are software engineers and the remaining two are other types of engineers. None were paid for their participation in the present study.

Materials

The materials consisted of a set of 20 algebraic word problems. These questions include concepts of basic algebra such as finding the perimeter of a polygon, setting up an equation with up to two different variables, and rate questions. The questions were created by a group of Algebra 1 teachers under the supervision of a former math content coordinator for a public school district.

Procedure

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All participants participated individually with a single experimenter. The participant was given the set of 20 word problems and asked to solve each one. Each participant was asked to write down the steps that he or she went through to solve the problem while simultaneously thinking out loud (cf. Ericsson and Simon, 1984) about the steps s/he was taking. The experimenter recorded the think aloud process. Each participant was given unlimited time to complete the 20 problems and was not allowed to use external resources to assist in problem solving.

Results

Each participant's problem solving protocol was analyzed for the processes used by that participant to solve the problem. The goal was to determine if there were general procedures that each participant used across problems. Once individual problem solving processes were determined, the experimenters looked for commonalities and differences across both math practitioner and student participants. This analysis was qualitative in nature.

For students, there was a general trend to start the problem solving process by creating a formula in terms of the variable to be solved. Then, the student would proceed, using standard algebraic principles to solve for the variable. This is very consistent with Anderson's ACT-R model. In other words, students had learned to proceduralize the concept of solving for a variable given an equation. This finding is not surprising as it is the standard method that is taught in schools and in textbooks. In cases involving applications of algebra to geometry, such as finding side lengths of triangle or rectangles that are expressed in terms of variables, students typically drew pictures of the geometric shapes and then labeled the sides before setting up their equations. Again, this seems fairly standard compared to what would be expected based on classroom instruction. One notable observation is that students rarely checked the correctness of their answers by plugging them back into the equations they set up to see if the equations balanced with the values they found.

In most instances, the students' process proved to be efficient at arriving at the correct answers. However, there were some notable drawbacks. First, the fact that students generally did not check their answers meant that if they used the incorrect initial equations or made a mistake in their calculations, they typically did not catch the mistakes and assumed the answers they got were correct. This also implies that, at least in some cases, they did not seem to have a sense of what a reasonable answer to the question would look like. Similarly, they were prone to errors when the answer they got was correct for a variable within the problem, but was not the answer the problem was asking for. For example, in some of the geometry-related problems, side lengths of a polygon were given in terms of the smallest side (e.g., two inches longer than the

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smallest side or twice as long). Students might set up an equation where each side is listed in terms of the smallest side (e.g., X, X+2, 2X) and solve for X. However, if the equation asked for a side other than the smallest side, the student might still give the value for X, since that was the value s/he solved for.

Second, if the initial equation that the student set up was incorrect, that often threw the student off. The student might then try a variation of the initial equation, rather than trying a completely different approach. Alternatively, students might try different combinations of the information given in the problem (such as adding or subtracting the numbers presented in the problem) to see what answers those produced.

It should also be noted that students were very consistent with each other in how they approached the initial setting up of their equations. This, again, is not surprising, since all were recruited from Northern Virginia schools and had probably been taught algebra in a similar fashion.

Math practitioners were much more diverse in their problem solving processes. In many cases, the methods they used were actually easier than those used by students. This may seem almost counterintuitive since much of the research in expert knowledge reports how much deeper and richer expert knowledge is than that of novices. Therefore, one might suspect that practitioners would use that deeper, richer knowledge to come up with more sophisticated problem solving techniques. Instead, the reverse was often the case.

For example, one of the problems given to the participants was to find two consecutive integers that add to 71. Students solved this problem by assigning the variables X and X+1 to the integers, adding them to get 2x+1=71, subtracting 1 to get 2x = 70, dividing both sides by 2 to get X = 35 and then adding 1 to get that the greater integer was 36. Practitioners typically solved this problem by dividing 71 by 2 to get 35.5 and then adding and subtracting .5 to 35.5 to get 35 and 36.

Another common strategy that math practitioners used was to categorize the problem types so that they could reuse strategies. It was common to hear in the session recordings statements such as “Oh, this is one of those consecutive integer problems where I divide by 2 and then ‘seesaw’ the number to get the two integers.” or “This is one of those problems with a fixed amount and a variable amount, so I’m going to subtract the fixed amount and divide by the coefficient of the variable.” This is very consistent with findings reported by Weiser and Shertz (1983) that experts categorize problems by the types of problem solving processes that will be used to solve them.

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Another process that practitioners engaged in was to think about constraints on what a reasonable answer might be. For example, it was noted that for problems that required the participant to find quantities like number of people or number of products sold, the answer would be an integer, whereas if the problem asked about a length of time, the answer could be fractional. Practitioners also gauged what would be a reasonable answer to real world problems based on what they knew about the world. This helped them validate whether the answer to their calculations looked reasonable.

Math practitioners also relied on heuristics to make the problems easier to solve. The above description of the process of finding consecutive integers that add to a number is one such heuristic. Another was to multiply decimals by powers of ten to get rid of the decimals to make computations easier. When asked to find the perimeter of a rectangle, which can be expressed as 2L (two times the length) * 2W (two times the width), practitioners typically found half the perimeter instead using L + W.

It should be noted that the heuristics used by experts were derivable from the formal algebraic methods. For example, the heuristic used to find two consecutive integers that add to 71 consisting of dividing 71 by 2 and then adding and subtracting .5 is derivable from the formal algebraic step of $2x + 1 = 71$. If we divide by 2 at that point, we get $x + .5 = 35.5$ and if we subtract the .5 as the practitioners did, we get 35 and if we add .5 to get x+1 on the left, we get 36 on the right.

While the general procedures used by practitioners to arrive at the actual answers to the problems varied, there were some common themes. The general process flowed as follows:

1. Read problem to pick out relevant information and what is being asked as the answer
2. Set up variables
3. Set up formula
4. Execute formula to get answer
5. Check answer to problem requirements, including plug back into equation

The heuristics and other strategies cited above were integrated into this overall process, often in service of simplifying what needed to be done. For example, the consecutive integer strategy was employed in the set up formula step of the above process. Real world considerations about what a reasonable answer might look like was integrated into the first step. Using previous

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problem types to define strategies involves using step 1 to recognize the similarities between the problems, step 2 to adapt the variables to the current problem, and step 3 to set up the formula to be solved to arrive at the answer.

It is worth noting that when problems were of a more traditional "algebraic" nature and did not depict a real world scenario, practitioners resorted to traditional algebraic methods. For example, one of the problems presented was "Five times the sum of eight and some number yields an amount equal to 140. What is the number?" In cases like this, both students and practitioners used the equation $5(8+x)=140$ to solve for the unknown number.

Overall, math practitioners spent more time than students did on initial analysis of the problem and used a more conceptual rather than procedural problem solving process. Both of these findings are consistent with findings reported by Leddo et al. (1990) that experts spend more time than novices in initial analysis of problems and approach problems more conceptually, rather than procedurally, when compared to novices.

Discussion

The results of the present project suggest that students tend to solve algebraic problems using formal algebraic methods, a process similar to a procedural method as described in Anderson's ACT-R framework. While generally efficient and accurate, this method can run into difficulties when procedures used are not appropriate for the given problem or the students make errors in computations. Then, they are often stuck or give an incorrect answer, thinking it is the correct one.

Math practitioners, on the other hand, tend to spend more upfront time analyzing problem features, the required answers, and information given before assigning variables and selecting problem solving methods. They use a more diverse range of problem solving approaches, all of which are derivable from formal algebraic methods, and often rely on simpler heuristics for solving problems than students do. Practitioners often check their answers for reasonableness, conformity for what the problem is asking and numerical accuracy by plugging the answers back into the formulas they use to solve the problems. Practitioners also recognize patterns and similarities across problems and use those patterns and similarities to help them select formulas to solve the problems. As such, math practitioners behave more like the experts studied in the Leddo et al. (1990) project in which experts were shown to be guided by mental models and other more conceptual reasoning strategies.

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The implication from the results of the present study is that mathematics instruction can be enhanced by going beyond the traditional formal problem solving processes that students are taught (and which are necessary to learn) and include instruction that enables students to build the abstract reasoning processes that math practitioners have.

Methods for developing abstract reasoning processes in students may be suggested by examining the problem solving world practitioners operate in compared to that in which students operate. Students learn math in courses, such as pre-algebra, algebra 1, geometry, etc. Each course is broken into units, e.g., linear equations, quadratics. Within each unit, students are told what problem types they will learn to solve and what formulas they will need to solve them. When they are given homework and unit tests, the problem types and formulas needed match what is covered in the lessons. Even though students do get midterm and final exams that cover multiple topics, they typically still fit within the neat framework of algebra, pre-algebra, etc.

Math practitioners operate in a different world. An actuary is asked to create a retirement plan or calculate what premiums should be charged for life insurance. There is no "formula" for determining what to charge people for life insurance. Answering that question takes into account a number of variables about the person being insured, the amount of the insurance, the cost structure of the insuring company and a host of other factors. In other words, an actuary cannot just jump into the problem, set up an equation and solve for X. The actuary has to think about what variables impact the answer, how they interact with each other, what are the unknowns and how they might affect the solution, etc. In other words, the actuary needs to do considerable upfront thinking and information gathering before s/he can even begin to set up formulas and perform calculations. It is only logical that, after spending years approaching professional problems this way, an actuary would approach real-world algebra problems using a similar process.

Similarly, the amount of data that actuaries (and other practitioners) deal with is enormous. The number of potential computations an actuary (and other practitioners) have to make is also enormous. It is logical that, as a practical matter, practitioners would look for ways to reduce complexity and simplify problem solving. It is also logical that practitioners would seek heuristics to simplify algebraic problem solving as well.

Finally, if practitioners are used to dealing with large, complex and time consuming problems, it is logical that they would look for similar patterns in the problems they solve in order to apply proven solutions for one problem to a similar problem. We see this professionally all the time. For example, lawyers use model contracts that they then adapt to specific clients' needs in order

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to reduce the time and effort required to draft a new contract. It is only logical that practitioners would apply the same thinking to algebraic problem solving.

The above analysis can be used to suggest teaching methods to build in students the kinds of conceptual problem solving skills that practitioners have. One method is to give students more experience working with ill-defined problems in real-world settings, similar to what math practitioners deal with in their everyday jobs. In schools, project-based learning (cf. Indrawan, Jalinus, and Syahril, 2018) can provide such an opportunity. In Northern Virginia, we have seen very limited use of project-based learning, and typically, the emphasis is on English or foreign languages. Math-related project-based learning assignments could involve such things as setting policy rates for life insurance (based on life expectancy, death benefit amount, corporate overhead and profit, etc.)

It was noted that practitioners typically rely heavily on heuristics. While formal methods are important to learn as the basis for deriving heuristics, we find very little systematic teaching of heuristics or how to derive them in mathematics classrooms. Sometimes, students report the opposite: their teachers insist on students solving problems using the methods the teachers taught with no room for individual creativity in the problem solving process. We propose, insead, that after teaching formal methods, teachers should teach students shortcuts and other heuristics while showing how they can be derived from the more formal methods they just learned.

Another strategy used by practitioners is recognizing problem solving patterns. Here students would learn different problem types or problems and the solutions associated with each. Then, when they see similar problem patterns, they will know how to solve them. We have experience working with this strategy. When working with students studying for the SAT, a test taken by students applying to college, Leddo, Sangela and Bekkary (2021) found that students who were taught the different patterns for circle problems found in the SAT scored higher than those who learned the formula-based approach to solving SAT circle problems published by the College Board, makers of the SAT test. A similar approach can be taken when teaching other math concepts.

Conclusion

The present study suggests that math practitioners employ a variety of problem solving processes and reasoning skills that math students do not have. These processes and reasoning skills are likely to have developed as a result of the practitioners' experience working with complex, ill-structured problems in their jobs. We hypothesize that students may be able to learn similar

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skills by replicating some of the problem solving conditions that practitioners face. Several potential techniques have been proposed. These should be the focus of future research and, if successful, be incorporated into the teaching of mathematics in K-12.

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